

行政院國家科學委員會專題研究計畫 成果報告

一對可交換方陣所生成的代數

計畫類別： 個別型計畫

計畫編號： NSC91-2115-M-032-003-

執行期間： 91 年 08 月 01 日至 92 年 07 月 31 日

執行單位： 淡江大學數學系

計畫主持人： 陳功宇

計畫參與人員： 劉鑠榮

報告類型： 精簡報告

處理方式： 本計畫可公開查詢

中 華 民 國 92 年 11 月 25 日

行政院國家科學委員會專題研究計畫成果報告

一對可交換方陣所生成的代數

The algebra generated by commuting pairs of matrices

計畫編號: NSC91-2115-M-032-003

執行期限: 91 年 8 月 1 日至 92 年 7 月 31 日

主持人: 陳功宇

執行機構及單位名稱: 淡江大學數學系

計畫參與人員: 劉鑠榮

執行機構及單位名稱: 淡江大學數學系

中文摘要

我們研究一對可交換方陣所生成的代數何時其維度恰是方陣的階

關鍵詞: 可交換方陣, 循環矩陣.

Abstract

Let A, B be $n \times n$ complex matrices, $\text{Alg}(A, B)$ be the algebra generated by A, B and identity matrix I . It is known that $\dim \text{Alg}(A, B) \leq n$, if $AB=BA$. We study that when $\dim \text{Alg}(A, B)=n$ for two commuting pairs of matrices A, B .

key words: commuting matrices, cyclic matrices, unispectral matrices.

1. Introduction

The algebra generated by two commuting matrices is, in particular, a vector space; what can its dimension be?

For matrices of order n over algebraically closed fields, it is known that the largest possible value of that dimension is equal to n . This dimension inequality was proved by Gerstenhaber [3], who made non-trivial use of some techniques of algebraic geometry. In 1990, Baria and Halmos [1] offer another proof, which use elementary methods only and which, moreover, concretely exhibits a vector basis of the generated algebra. It suggests that we consider the problem when $\dim \text{Alg}(A, B)=n$ for two commuting pairs of matrices A and B . The problem is difficult, we only have some partial results.

A unispectral matrix is a matrix with only one eigenvalue. A cyclic matrix is a matrix that has a cyclic vector.

Every unispectral matrix is similar to a direct sum of Jordan blocks. A Jordan block of size n with λ , denoted by $J(n, \lambda)$, is a square matrix such as is exemplified by

$$J(n, \lambda) = \begin{bmatrix} \lambda & 1 & 0 & \Lambda & 0 \\ 0 & \lambda & 1 & 0 & M \\ M & 0 & \lambda & 0 & 0 \\ M & & 0 & 0 & 1 \\ 0 & \Lambda & \Lambda & 0 & \lambda \end{bmatrix} ..$$

The largest number of cyclic direct summands that occurs in a unispectral direct summand of the Jordan form is called the height. Our result is the following

2. Main results

Theorem 1. Let A be a unispectral matrix with size $2m$ and height 2. If the degree of the minimal polynomial of A is m and $BA=AB$, then

$\dim \text{Alg}(A, B) < 2m$ if and only if

B is unispectral and $A^{m-1}(\lambda - B) = 0$ for the eigenvalue of B.

Theorem 2. Let $A = J(m, \lambda) \oplus \lambda I_k$, where I_k is the identity matrix of size k .

Let $P = O \oplus I_k$ and $BA=AB$. Then

(a) $\dim \text{Alg}(A, B) = m + k \Rightarrow PBP : \text{Range}(P) \rightarrow \text{Range}(P)$ is cyclic.

(b) B is nonunispectral and

$PBP : \text{Range}(P) \rightarrow \text{Range}(P)$ is cyclic $\Rightarrow \dim \text{Alg}(A, B) = m + k$.

The following example show that if B is unispectral, then Theorem 2.(b) is not true.

Example:

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $AB=BA$ and $AC=CA$.

It is easy to see that $\dim \text{Alg}(A, B) = 4$ but $\dim \text{Alg}(A, C) = 3$.

參考資料:

- [1] J.Barrio and P.R.Halmos, Vector bases for two commuting matrices, LAMA 27(1990),147-157.
- [2]M.Gerstenhaber, On dominance and varieties of commuting matrices, Annals of Math.73(1961),324-348.
- [3]R.M.Guralnick, A note on commuting pairs of matrices, LAMA 31(1992),71-75.
- [4]T.Laffey and S.Lazarus, Two-generated commutative matrix subalgebras, Linear

Alg. and Appl. 147(1991)249-273.

[5]A.Wadsworth,On commuting pairs of matrices,LAMA27(1990),159-162.